ETMAG

CORONALECTURE 5

Asymptotes,

Continuous functions,

Differentiation

April 20, 12:15

Definition.
The line x = a is a *vertical asymptote* of a function y = f (x) iff

$$\lim_{x\to a^+} f(x) = \pm\infty \text{ or } \lim_{x\to a^-} f(x) = \pm\infty$$

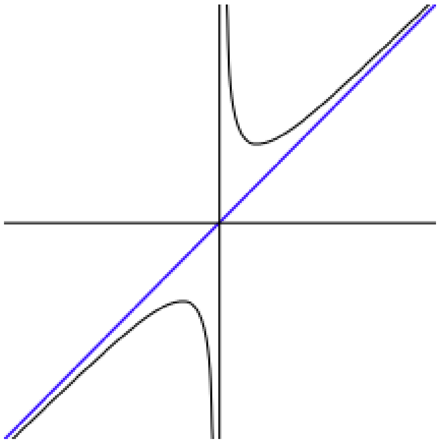
Definition.
The line y = ax+b is an *oblique asymptote* of the (graph of a) function y = f (x) iff

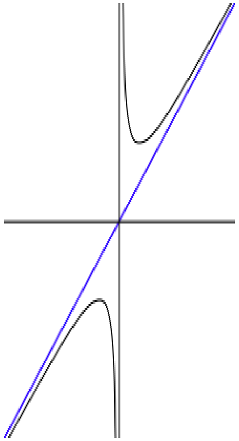
$$\lim_{x\to +\infty} [f(x) - (ax+b)] = 0 \text{ or } \lim_{x\to -\infty} [f(x) - (ax+b)] = 0.$$

When a = 0 the oblique asymptote is called the *horizontal asymptote*.

Essentially, we should say *asymptote* ***of the graph*** *of a function* because it is a geometrical object.

Example.





Lines x=0 and y=x are vertical and oblique asymptotes, respectively, for $f(x) = \frac{1}{x} + x$,

Lines x=0 and y=2x are vertical and oblique asymptotes for $f(x) = \frac{1}{x} + 2x$, resp.

FAQ. How the hell do I find an asymptote for f(x)?

To find vertical asymptotes just look for points around which the values of your function are unbounded (division by zero, logarithm close to zero and the like). To find an oblique asymptote look at the definition: the line $y = ax+b$ is the oblique asymptote for f(x) if and only if

$\lim_{x\to+\infty}[f(x) - (ax + b)] = 0$. Dividing both sides by x we get

$0 = \lim_{x\to+\infty}\frac{f(x)-(ax+b)}{x} = \lim_{x\to+\infty}\frac{f(x)}{x} - \lim_{x\to+\infty}\frac{ax}{x} - \lim_{x\to+\infty}\frac{b}{x} = \lim_{x\to+\infty}\frac{f(x)}{x} - a$,

hence, $\lim_{x\to+\infty}\frac{f(x)}{x} = a$. Once we have a, we check if there exists the limit $\lim_{x\to+\infty}(f(x) - ax)$. If it does then $b = \lim_{x\to+\infty}(f(x) - ax)$. If it does not – there is no asymptote at $+\infty$.

Theorem.

A function f(x) has an asymptote at $+\infty$ iff there exist limits $\lim\limits_{x\to+\infty} \frac{f(x)}{x} = a$ and $\lim\limits_{x\to+\infty} (f(x) - ax) = b$. Then the line $y = ax+b$ is the asymptote.

Notice that the existence of $\lim\limits_{x\to+\infty} \frac{f(x)}{x}$ is not enough. For example consider $f(x) = \sqrt{x}$. $\lim\limits_{x\to+\infty} \frac{\sqrt{x}}{x} = 0 = a$ and $\lim\limits_{x\to+\infty} (\sqrt{x} - 0x)$ does not exist.

A similar theorem is valid for an oblique asymptote at $-\infty$.

Switching to the old presentation for the

definition and properties of

CONTINUOUS FUNCTIONS.

-*Please, remind me to switch back here <u>before</u> we begin derivatives.*

-*But how will we know you are going to begin derivatives?*

-*Don't ask stupid question!*

-*There are no stupid questions, but there are many stupid requests …*

-*Aw, just shut up and do what you are told. Or else …*

Remark.

It is wrong to think that continuous functions are those whose graphs look continuous. The graph of a continuous function should look like an unbroken curve only if considered **on an interval contained in its domain**. For example, the *signum* function (1 if $x>0$, -1 if $x<0$) is continuous in its domain (reals without 0) even though its graph has a discontinuity. For the same reason *tan* is considered continuous. Even outrageously discontinuous functions, like the Dirichlet function, are continuous if you restrict them to a subset of $\mathbb{R}$, for example, it is continuous on $\mathbb{Q}$ (well, it is constant there).

Definition.
For every set A, A $\subseteq \mathbb{R}$, the *least upper bound* of A is the number *sup*(A) defined as the smallest number *t* such that every element of A is less than or equal to *t*.

Definition. (A twin to the *sup* definition)
For every set A, A $\subseteq \mathbb{R}$, the *greatest lower bound* of A is the number *inf*(A) defined as the largest number *t* such that every element of A is greater than or equal to *t*.

Theorem.
Every subset A of $\mathbb{R}$ bounded from above has the least upper bound (not necessarily belonging to A).

Similar theorem holds for sets bounded from below.

Theorem.
If f is continuous on a closed interval $[a;b]$ then it takes on its largest and its smallest values on the interval. Meaning there exists $x_{max} \in [a;b]$ such f(x_{max}) is the largest value of f on $[a;b]$.

Outline of proof.
Obviously the set f($[a;b]$) is bounded, so it has the least upper bound, say y_{max}. The problem is to prove that there exists x_{max} in $[a;b]$ such that f(x_{max}) = y_{max}. This is beyond the scope of this course.

The theorem may be rephrased as:

If f is continuous on a <u>closed</u> interval [a;b] then f([a;b]) is also a <u>closed</u> interval.

An illustration of the principle.

$f(x) = \frac{1}{x}$ is continuous on (0;1) but not on [0;1]. The set of values is unbounded from above so f(x) does not take its largest value. On the other end, it is bounded from below by 1 and the set of lower bounds has the largest element, namely 1. But 1 is not the value for this function for any point in the open interval (0;1). It is, of course, the value of f(x) for x=1 which does not belong to (0;1) but does to [0;1].

Comprehension.

What is tan([0;π])?

Switching to old presentation here for definition and properties of the derivative.

Back after we list properties of differentiable functions.

Proof (continuity of differentiable functions)

If a function $y = f(x)$ is differentiable at x_0 then it is continuous at x_0.

Since $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$ and $\lim_{h\to 0} h = 0$, we have

$$\lim_{h\to 0} (f(x_0+h) - f(x_0)) =$$

$$= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} h = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \lim_{h\to 0} h =$$

$$= f'(x_0)\cdot 0 = 0$$

which means

$$\lim_{h\to 0} f(x_0+h) = f(x_0) \quad \text{i.e.,} \quad \lim_{x\to x_0} f(x) = f(x_0).$$

Proof (of the product rule)

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x).$$

$[f(x)g(x)]'$ is the limit of the difference quotient

$$\lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} =$$

$$\lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} =$$

$$\lim_{h\to 0} \frac{f(x+h)(g(x+h)-g(x))+(f(x+h)-f(x))g(x)}{h} =$$

$$\lim_{h\to 0} \left(\frac{f(x+h)(g(x+h)-g(x))}{h} + \frac{(f(x+h)-f(x))g(x)}{h}\right) =$$

$$\lim_{h\to 0} \left(f(x+h) \frac{(g(x+h)-g(x))}{h} + \frac{(f(x+h)-f(x))}{h} g(x)\right) =$$

$$\lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{(g(x+h)-g(x))}{h} + \lim_{h\to 0} \frac{(f(x+h)-f(x))}{h} \lim_{h\to 0} g(x) =$$

$$f(x)g'(x)+f'(x)g(x)$$

Proof. (Derivative of the inverse function)

$(f^{-1})'(y) = \frac{1}{f'(x)}$ where $y = f(x)$.

First look at this $f^{-1}\big(f(x)\big) = x$. Differentiating both sides we get

$\Big(f^{-1}\big(f(x)\big)\Big)' = 1$ hence, by the chain rule

$(f^{-1})'\big(f(x)\big)f'(x) = 1$, hence

$(f^{-1})'\big(f(x)\big) = \frac{1}{f'(x)}$

or, if you denote $f(x)$ by y,

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

Derivative of the inverse function in action.

Let's calculate $(\arcsin x)'$, $x \in$ [-1;1]

$(\text{arc}\sin x)' = \frac{1}{(\sin t)'} = \frac{1}{\cos t}$, provided sin t = x and $t \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$.

Since $cos^2\, t + sin^2\, t = 1$ we have $|\cos t| = \sqrt{1 - sin^2\, t}$.

Since cos t > 0 on $\left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$, we can write $\cos t = \sqrt{1 - sin^2\, t} = \sqrt{1 - x^2}$.

Finally

$$(\text{arc}\sin x)' = \frac{1}{\cos t} = \frac{1}{\sqrt{1-x^2}}$$